TEACHERS BECOME INVESTIGATORS OF

Students' ideas about math

Learning involves seeing how children think about numbers

BY DEBORAH SCHIFTER AND MARGIE RIDDLE

As part of her morning routine, Margie Riddle had given her 4th graders a set of subtraction problems. Included among them were 145 – 100 and 145 – 98. As the children began to consider the latter problem, Riddle realized it provided an opening with great learning potential, so she deferred discussion until the math lesson later in the day.

What Riddle saw was this: Many of the children realized there was a connection between the two problems. However, after calculating 145 – 100 = 45 and before solving 145 – 98, they weren't sure if the answer to the second problem would be two more or two less than 45. Once they solved it and they knew the answer was 47, they wondered, why was it two more? After all, when they changed 100 to 98, they subtracted two, so why add two to get the right result? Their puzzlement, Riddle felt, could lead her students to a deeper appreciation of the meaning of the operation of subtraction.

When the class returned to this question later that day, Riddle's students began as they often did by sharing a variety of ways to calculate 145 – 98. But at one point, Brian insisted on explaining his thinking to his classmates. "It goes with the problem before," he said. "It's like you've got this big thing to take away and then you have a littler thing to take away, so you have more. Can I draw a picture?" He went to the blackboard, thought for awhile, and then drew a big blob. (See illustation on page 30.)

Riddle described what happened next. "Brian's classmates were watching and listening fairly intently, and

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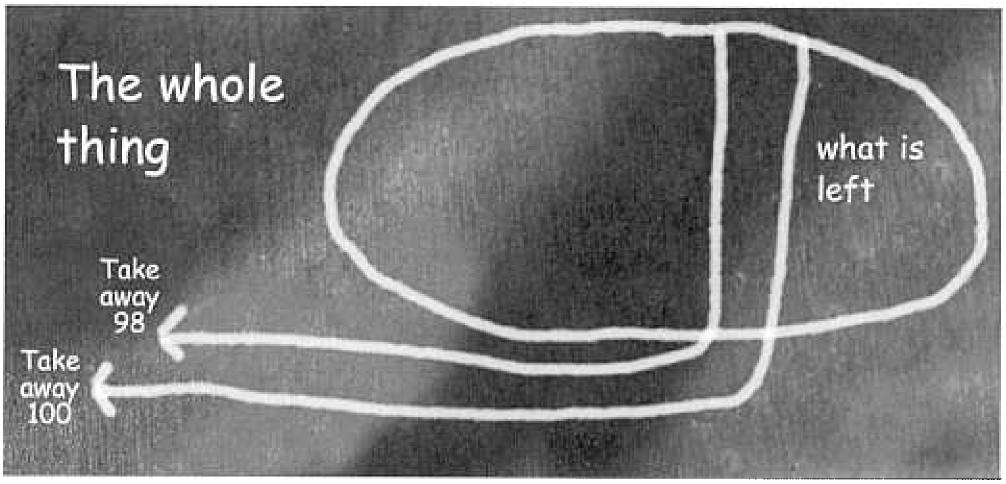
suddenly, inspired by his presentation, Rebecca said excitedly, 'Yeah, it's like you have this big hunk of bread and you can take a tiny bite or a bigger bite. If you take away smaller, you end up with bigger.'" Riddle asked if Rebecca thought this would always be true, and Rebecca said, "I think so."

Max had been quiet, but inspired by Rebecca's explanation and Brian's picture, he carried the thinking further: "Yeah, the less you subtract, the more you end up with. And," he continued with emphasis, "the thing you end up with is exactly as much larger as the amount less that you subtracted."

THE LESSON IN THE LESSON

Riddle's behavior may puzzle those who remember the math classes of their childhood. Mathematics has been taught the same way for decades: The teacher shows students procedures for getting right answers and then monitors students as they reproduce those procedures. Why does Riddle spend so long having her students discuss such a simple problem as 145 – 98? Once students agree that the answer is 47, why doesn't she just move on?

Riddle's behavior becomes comprehensible when viewed in terms of what "doing" mathematics can mean, how learning takes place, and what these ideas imply for teaching mathematics. She acts on the belief that math is an interconnected body of ideas to explore, rather than a set of facts, definitions, and procedures that students must memorize and call up on demand. To "do" mathematics is to conjecture — to invent and extend ideas about mathematical objects — and to test, debate, and revise or replace those ideas. So when her students raised a question that went beyond merely finding the answer to 145 – 98, to exploring the relationship between two subtraction problems, Riddle realized she had an opportunity to mine the potential for powerful insights into the nature of the operation. A simple subtraction problem led this class into the territory of early algebra.



Brian said: "It's like you've got this big thing to take away and then you have a littler thing to take away, so you have more. Can I draw a picture?" He went to the blackboard, thought for awhile, and then drew a big blob.

In listening carefully to her students, taking in the content of their ideas, Riddle attended to their individual process at the same time she managed the group. At the end of the lesson, Riddle asked her students to write about what they discussed, allowing her to assess each student's understanding of the material.

WHAT TEACHERS MUST KNOW

In a teaching practice like Riddle's, the teacher's responsibilities extend far beyond demonstrating the next algorithm students must memorize. Instead, by setting challenging, often open-ended problems, teachers elicit and build on their students' mathematical insights as well as their questions. If teachers themselves have been taught mathematics as discrete procedures and definitions to be memorized, how can they prepare to take on such an approach? Part of the answer lies in new forms of professional development.

Teachers teach according to some conception — most often implicit in what they do, less often explicitly held · of how learning takes place and of the nature of the content they teach. While this may seem obvious, it is a starting point for teacher development: Challenge those conceptions and open them up to reflection. With the occasion teachers then have to gain deeper insight into instructional theory and practice, new visions can begin to emerge.

If these insights are to result in significant and enduring change in the way teachers teach, they cannot be induced by lectures or workshops. Teacher development programs must dig deeper, giving participants opportunities to construct for themselves more powerful understandings of learning, teaching, and disciplinary substance.

DIG INTO CONTENT

A first step in helping teachers envision a new practice is to make them students in classrooms in which this practice is used. Rather than trying lessons designed for children, however, teachers must be challenged at their own level of mathematical competence, confronted with problems and concepts they have not encountered before.

Such courses must address the content teachers teach. If teachers are to teach mathematics differently from the way they themselves were taught, they must know the mathematics of their own curriculum. Although it is routinely assumed that school mathematics, particularly at the elementary level, is so simple that any educated adult knows it well enough to teach it, this is not the case (Conference Board of the Mathematical Sciences, 2001). The mathematics is, in fact, conceptually complex but rarely taught in the United States in ways that acknowledge this. Elementary teachers need to understand the structure of the base-10 number system, the meaning of the basic operations, the logic of rational numbers, and the properties of geometric shapes.

Secondary teachers must go below the surface of algebraic manipulation to consider the relationships that are expressed, recognize how features of functions appear in different forms of representation, and understand the nature of proof.

At any level, teachers must develop richly connected mathematical concepts. Teachers will need the connections among concepts and their contexts and the ability to use them flexibly and creatively to teach mathematics well.

In Riddle's case, she needed as a 4th-grade teacher to have the numerical flexibility her students are developing and, beyond knowing standard algorithms for subtraction, had to understand the situations the operations modeled and be able to call up contexts to illustrate the concepts. She needed to recognize the value of exploring relationships between arithmetic expressions and the power of articulating generalizations.

With that knowledge, Riddle is positioned to structure her classroom around her students' mathematical thinking. Her 4th graders have opportunities to pose their own questions, to articulate and analyze their own and their classmates' reasoning and, in doing so, to deepen their understanding.

Similarly, when teachers are students in mathematics classrooms, they must have opportunities to pose their own questions, to articulate and analyze their own and other teachers' reasoning, and to deepen their understandings. In this type of professional learning situation, the instructor pauses the mathematics lesson periodically, allowing teachers to discuss or write about what they are experiencing. Thus, these mathematics lessons become occasions for teachers to reflect on their own process of learning and to consider those features of the classroom that support or hinder that process. All of this may inspire teachers to envision a new kind of practice for themselves and help prepare them to put it into play

In this way, teachers develop new conceptions of the nature of mathematics and a heightened sense of their own mathematical powers. As learners of mathematics, they experience a new kind of classroom, one in which the social character of doing mathematics is realized, and in which student thinking takes center stage.

Professional development support

To create a practice based on conceptual understanding of mathematics, teachers must:

- Envision a new kind of classroom.
- Develop deeper understanding of mathematics.
- Learn to listen to and engage with students' mathematical thinking.

Professional development activities to support teachers in developing a new practice include:

- Teaching teachers mathematics while modeling practice.
- Using classroom cases to analyze student thinking and examine the teacher's role.
- Having teachers write cases that capture their own students' mathematical thinking.

DEVELOP A LISTENING EAR

Knowing mathematics more deeply and developing a vision for teaching based on that deeper understanding provide a start. Yet Riddle's practice calls for more. The possibility of engaging students in exploring ideas presupposes being able to identify the concepts that are at issue for them. This is no simple task since their ideas are expressed in the words of young beginners. Teachers need to develop the skills of listening to students' words, interpreting the mathematical ideas they express, and identifying the relationship between student thinking and the mathematics on her agenda. Such a practice involves developing a new ear, one that is attuned to the mathematical ideas of one's own students. Teachers who began to listen to their students in new ways commented:

"I [used to] listen for right answers, confirmation that the students understood what they had been taught. I was accustomed to listening for specific indicators that a student was following my line of thinking" (Natowich, 1992).

"It is almost as if I heard them previously, but I had my next statements already planned. I attempted to adjust their thinking to what I planned to say next, instead of analyzing what they said to determine what I should ask, say, or do next" (personal communication, 1992).

In a practice that puts students' ideas at the center of instruction, teachers must listen to students not only to assess the extent to which those students' ideas match their own, but also more importantly to understand these ideas in their own right. What is it that this child is communicating? What is the sense in this child's idea? What does he/she understand?

When a child offers incorrect or incomplete ideas, the teacher must ask: What is the child confused about? What is the issue the child is working on? How does it connect to the content I am to teach?

One way to help teachers develop an ear for students' mathematical ideas is to study and discuss classroom cases that present children's articulation of their thinking. Although each case features specific children in specific classrooms, carefully selected cases support insights of a more general nature. They allow teachers to develop the skill of hearing the mathematics in what a child is saying, following a student's mathematical reasoning, identifying conceptual issues with which particular children are occupied, and situating these issues in a larger mathematical frame of reference. (For a source of cases, see the *Developing*

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Mathematical Ideas curriculum; Bastable, Schifter, & Russell, 2002; Russell, Schifter, & Bastable, 2002; Schifter, Bastable, & Russell, 1999a, 1999b, 2002.)

Published cases of other educators working with children provide opportunities for discussion and exploration, but teachers also need to learn to listen and to hear their own students making sense of mathematics. Writing cases that capture the thinking of one or more of one's own students is effective, as in Riddle's case example that she wrote. Teachers choose different methods for recording student dialogue — audio recording, taking notes during class, or writing down what happened as soon after the lesson as possible — and then write a narrative based on that dialogue, including their thoughts, questions, and decisions. No matter which method teachers use, they report that the exercise has a powerful effect on their practice. It forces them to examine closely their students' words and helps them work out what understanding, questions, and confusion lie behind those words. As teachers themselves have reported:

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"Writing [cases] forces me to listen more carefully. I must understand what students are saying and thinking to be able to write the episode. My writing preserves details that I forget with the passage of time. The [cases] capture moments of confusion and deep thinking that have resulted in students becoming more open and articulate" (personal communication, 1995).

"Writing the [cases] became a window into my students' thinking. I've talked about listening to my students in the past as a personal goal and responsibility of mine, but the [cases] forced me to really listen to them. What were their words? What did I make of their explanations? What does it tell me about their mathematical thinking?" (personal communication, 1995).

In this way, teachers' own classrooms become a resource for learning about student thinking.

Exploring mathematics, analyzing cases of student thinking, and writing cases from one's own teaching all help participants work through content areas and develop skills required to support students' conceptual development. These areas also help teachers develop an inquiring disposition. The new understanding they must develop and the teaching situations they must negotiate are too varied, complex, and context-dependent to be anticipated in one or even several courses. Teachers must become investigators — of mathematics and of student thinking — in their own classrooms.

The goal of teaching mathematics for conceptual understanding entails an instructional practice that treats mathematics as a realm of ideas to be explored rather than exclusively a set of facts, procedures, and definitions to be memorized. In this way, we provide students with the capability to reason mathematically and to solve the new kinds of problems they will inevitably face in the future (National Council of Teachers of Mathematics, 2000).

RESOURCES

Developing Mathematical Ideas is a professional development program for teachers based on the principles described in this article. See www2.edc.org/CDT/dmi/dmicur.html.

Lenses on Learning is a professional development program for administrators based on the principles described in this article. See www2.edc.org/CDT/cdt/cdt_admin.html.

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